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Pointwise Simultaneous Approximation by Rational Operators

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We obtain pointwise simultaneous approximation estimates for rational operators which are not possible by algebraic polynomials. © 1991 Academic Press, Inc.

1. INTRODUCTION

Timan [11, Theorem 1, p. 17] proved that, for every $n = 1, 2, 3, \ldots$, if $f \in C^q(I)$, where $I = [-1, 1]$, then there exist polynomials P_n of degree n such that for $x \in I$

$$|f(x) - P_n(x)| = O\left\{\left(\frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right)^q \omega\left(f^{(q)}; \frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right)\right\}, \quad (1.1)$$

where $\omega(f; \delta)$ is the usual modulus of continuity of f.

It follows from (1.1) that the rate of convergence to the function f by algebraic polynomials is faster near the endpoints.

Later Trigub obtained an extension of Timan's theorem in [12, Lemma 1, p. 263], by proving that for every $n = 1, 2, 3, \ldots$, there exist polynomials P_n such that

$$|f^{(k)} - P_n^{(k)}(x)| = O\left\{\left(\frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right)^{q-k} \omega\left(f^{(q)}; \frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right)\right\}$$

for $x \in I$ and $k = 0, \ldots, q$.

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The following question was posed by Kolmogorov [5, p. 163]: can one have in formula (1.1) a faster rate of convergence to the function f by algebraic polynomials at the endpoints? In other words, can we replace the expression $\sqrt{1-x^2}/n + 1/n^2$ by $o(\sqrt{1-x^2}/n) + o(1/n^2)$ in (1.1)?

Telyakovskii [10, Theorem 2, p. 259] and Gopengauz [5, Theorem 1, p. 168] gave the following refinement of Timan's theorem: if $f \in C^q(I)$, then, for $n \geqslant 4q+5$, there exist polynomials P_n such that

$$|f^{(k)}(x) - P_n^{(k)}(x)| = O\left\{\left(\frac{\sqrt{1-x^2}}{n}\right)^{q-k} \omega\left(f^{(q)}; \frac{\sqrt{1-x^2}}{n}\right)\right\}$$

for $x \in I$ and for $k = 0, ..., q$.

Gopengauz in [5, Proposition 2, p. 165] proved that $(1-x^2)^{1/2}$ cannot be replaced by $(1-x^2)^{\beta}$ with $\beta > \frac{1}{2}$ in (1.1). Moreover, he noted in [5, Proposition 1, p. 164] that the endpoints are special points; that is, it is not possible to have analogous estimates to (1.1) for some interior point a. In fact, for $|a| < 1$, Gopengauz proved in the same paper that there exist continuous functions f on I for which there are no algebraic polynomials P_n of degree less or equal n such that

$$|f(x) - P_n(x)| = O\left\{\omega\left(f; \frac{\sqrt{1-x^2}}{n}\, \varepsilon(1-x^2) + \frac{\delta(n^{-1})}{n^2}\right)\right\} \qquad (1.2)$$

and

$$|f(x) - P_n(x)| = O\left\{\omega\left(f; \frac{\varepsilon(|x-a|) + \delta(n^{-1})}{n}\right)\right\} \qquad (1.3)$$

for all integers n and $x \in I$, where $\varepsilon(u) \downarrow 0$ and $\delta(u) \downarrow 0$, when $u \to 0$.

The main aim of this paper is to show that it is possible to obtain estimates of type (1.2) and (1.3) by rational approximation. In addition, we give pointwise simultaneous approximation estimates for such rational approximation.

2. Main Results

Let $Y = (y_{n,i} \in I: i = 0, ..., n;\ n \in N)$ be an infinite matrix where each row is a set of distinct points in I and define for $f \in C^r(I)$ the operator

$$S_{n,r}^{s}(Y; f; x) = \frac{\sum_{k=0}^{n} \left(|x - y_{n,k}|^{-s} \sum_{j=0}^{r} f^{(j)}(y_{n,k})/j!\,(x - y_{n,k})^j\right)}{\sum_{k=0}^{n} |x - y_{n,k}|^{-s}}$$

with $s = r + \alpha$, where $\alpha > 1$ is fixed.

Note that for $\alpha = 4$, $S^s_{n,r}$ coincides with Szabados' operator [9, p. 220]. If $r = 0$, $S^s_{n,r}$ coincides with the Shepard-Balázs operator widely used in approximation theory and in fitting data, curves and surfaces [1, 2, 6–8]. The extension of $S^s_{n,r}$ to several dimensions was also investigated by Farwig [4, p. 578].

Obviously, if $\alpha \in N$, then $S^s_{n,r}$ is a rational linear operator of degree $(r + sn, sn)$. $S^s_{n,r}$ preserves polynomials of degree r, i.e.,

$$S^s_{n,r}(Y; e_i; x) = e_i, \qquad e_i(x) = x^i, \qquad i = 0, ..., r.$$

It also satisfies

$$D^i S^s_{n,r}(Y; f; y_{n,k}) = f^{(i)}(y_{n,k}), \qquad k = 0, ..., n, \; i = 0, ..., r$$

and

$$D^i S^s_{n,r}(Y; f; y_{n,k}) = 0, \qquad k = 0, ..., n, \; i = r + 1, ..., [s] - 1.$$

Following a procedure given in [3], we can write

$$S^s_{n,r}(Y; f; x) = H_n(Y; f; x) + R_n(Y; f; x),$$

where $H_n(f)$ is a polynomial such that

$$D^i H_n(Y; f; y_{n,k}) = f^{(i)}(y_{n,k}), \qquad i = 0, ..., r, \; k = 0, ..., n$$

and $R_n(f)$ is a rational expression such that

$$D^i R_n(Y; f; y_{n,k}) = 0, \qquad i = 0, ..., r, \; k = 0, ..., n.$$

In other words, $S^s_{n,r}$ can be considered as a rational extension of the Hermite interpolating polynomial.

We want to use the rational operator $S^s_{n,r}$ to obtain pointwise estimates which are not possible with polynomial approximation. Given $p = 1, 2, 3, ...$, let $x = x_p\colon [0, 1] \to [-1, 1]$ be the function defined by

$$x = x(\theta) = \begin{cases} (2\theta)^{2p+1} - 1, & \theta \in [0, \frac{1}{2}] \\ -(2 - 2\theta)^{2p+1} + 1, & \theta \in [\frac{1}{2}, 1] \end{cases} \tag{2.1}$$

and consider the matrix $X = (x_{n,k} = x(k/n)\colon k = 0, ..., n, \; n \in N)$. Then we have

THEOREM 2.1. *For all $f \in C^r(I)$ and for all $i = 0, ..., r$,*

$$|[f(x) - S^s_{n,r}(X; f; x)]^{(i)}| \leqslant A \frac{[(1 - x^2)^{2p/(2p+1)}]^{r-i}}{n^{s-i-1}} \int_{n^{-1}}^{1} \omega(f^{(r)}; t(1 - x^2)^{2p/(2p+1)}) \frac{dt}{t^{s-r}}, \tag{2.2}$$

where $x \in I$, $s = r + \alpha$, $\alpha > 1$, *and* A *is a constant depending only on* $p, r,$ *and* α.

Inequality (2.2) generalizes and improves the bound obtained in [3]

$$|f(x) - S^2_{n,0}(X^*; f; x)| \leqslant \text{const}\, n^{-1} \int_{n^{-1}}^{1} \omega(f; t\sqrt{1-x^2})\, t^{-2}\, dt,$$

where $f \in C(I)$ and

$$X^* = \left(x_{n,k} = \cos \frac{k\pi}{n} : k = 0, ..., n\right).$$

From Theorem 2.1 we have

COROLLARY 2.2. *For all* $f \in C^r(I)$ *and for all* $i = 0, ..., r$,

$$|[f(x) - S^s_{n,r}(X; f; x)]^{(i)}| \leqslant A \frac{[(1-x^2)^{2p/(2p+1)}]^{r-i}}{n^{r-i}} \omega\left(f^{(r)}; \frac{(1-x^2)^{2p/(2p+1)}}{n}\right), \tag{2.3}$$

where $x \in I$, $s = r + \alpha$, $\alpha > 1$, *and* A *is a constant depending only on* $p, r,$ *and* α.

Inequality (2.3) is an estimate of type (1.2), as we mentioned in the introduction. Now we may ask if the endpoints 1 and -1 are special points or whether there is an analogous result to (2.2) for some interior point. The latter possibility was already suspected by Criscuolo and Mastroianni in [3]. We now show that this is indeed the case. Setting

$$Z = \left(z_{n,k} = \left(\frac{2k}{n} - 1\right)^{2p+1} : p \in N, k = 0, ..., n, n \text{ even}\right),$$

we have

THEOREM 2.3. *For all* $f \in C^r(I)$ *and for all* $i = 0, ..., r$,

$$|[f(x) - S^s_{n,r}(Z; f; x)]^{(i)}| \leqslant A \frac{[|x|^{2p/(2p+1)}]^{r-i}}{n^{s-i-1}} \int_{n^{-1}}^{1} \omega(f^{(r)}; t\,|x|^{2p/(2p+1)}) \frac{dt}{t^{s-r}}, \tag{2.4}$$

where $x \in I$, $s = r + \alpha$, $\alpha > 1$, *and* A *is a constant depending only on* $p, r,$ *and* α.

From Theorem 2.3 we have

COROLLARY 2.4. *For all $f \in C^r(I)$ and for all $i = 0, ..., r$,*

$$|[f(x) - S^s_{n,r}(Z; f; x)]^{(i)}| \leqslant A \frac{[|x|^{2p/(2p+1)}]^{r-i}}{n^{r-i}} \omega\left(f^{(r)}; \frac{|x|^{2p/(2p-1)}}{n}\right) \tag{2.5}$$

where $x \in I$, $s = r + \alpha$, $\alpha > 1$ and A is a constant depending only on p, r, and α.

Inequality (2.5) is an estimate of type (1.3), as in the introduction. As an application, let $0 < \beta < 1$ and $y_0, ..., y_q$ be $q + 1$ distinct points from I. We can consider the Shepard–Szabados operator $S^s_{n,r}$ corresponding to the matrix $X = (\phi(qk/n) : k = 0, ..., n,\ n = mq;\ n, m, q \in N)$, where

$$\phi(\theta) = \phi_i(\theta) = \begin{cases} \dfrac{y_{i+1} - y_i}{2} (2(\theta - i))^{2p+1} + y_i, & \theta \in [i, i + \frac{1}{2}] \\ \dfrac{y_i - y_{i+1}}{2} (2(i + 1 - \theta))^{2p+1} + y_{i+1}, & \theta \in [i + \frac{1}{2}, i + 1], \\ 0, & \text{otherwise} \end{cases}$$

for $i = 0, ..., q - 1$ and $p \in N$ is such that $\gamma := 2p/(2p + 1) > \beta$. As in the proof of Theorem 2.1, we obtain

COROLLARY 2.5. *For all $f \in C^r(I)$ there exists a rational function r_n such that for all $x \in I$*

$$|f(x) - r_n(x)| \leqslant A \left(\frac{d^\gamma(x)}{n}\right)^r \omega\left(f^{(r)}; \frac{d^\gamma(x)}{n}\right), \tag{2.6}$$

where A is a constant independent on n and f and $d(x) = \min_{i=0,...,q} |x - y_i|$ and $0 < \gamma < 1$.

Finally we remark that the exponent γ in (2.6) cannot equal 1; i.e., γ must be always less than 1. To see this, assume that there is a constant C with the property that for every f and every n there is a rational function r of degree at most (n, n) such that

$$|f(x) - r(x)| \leqslant C\omega\left(f; \frac{x}{n}\right), \qquad x \in [0, 1]. \tag{2.7}$$

Here, for convenience, we take $[0, 1]$ as the interval of approximation and not $[-1, 1]$ as before. Consider the function g defined by

$$g(x) = \begin{cases} 12x - 6, & \frac{1}{2} \leqslant x \leqslant 3/4 \\ -12x + 12, & \frac{3}{4} \leqslant x \leqslant 1, \end{cases}$$

and define the function f such that $f = g$ on $[\frac{1}{2}, 1]$ and

$$f\left(\frac{y}{2}\right) = \frac{f(y)}{2}, \qquad y \in [\tfrac{1}{2}, 1]$$

and, proceeding in this way step by step, define f on $[0, 1]$. Obviously, $f \in \mathrm{Lip}\, 1$ and $f(0) = 0$; therefore, from (2.7), we get for large enough n

$$|f(x) - r(x)| \leqslant x.$$

Since $r(0) = 0$, it follows that $r(x) = xr^*(x)$ with r^* a rational function. Now let $x = y/2^l$, $y \in [0, 1]$, and $l \in N$. Then

$$\left| f\left(\frac{y}{2^l}\right) - \frac{y}{2^l} r^*\left(\frac{y}{2^l}\right) \right| \leqslant \frac{y}{2^l}.$$

But, from the construction of f, it follows that $f(y/2^l) = f(y)/2^l$, so

$$\left| f(y) - yr^*\left(\frac{y}{2^l}\right) \right| \leqslant y.$$

Hence, letting $l \to +\infty$, we get for every $y \in [0, 1]$

$$|f(y) - c_1 y| \leqslant y \tag{2.8}$$

for some constant c_1. But, applying (2.8) for $y = 1$ and $y = \frac{3}{4}$, we have $|c_1| \leqslant 1$ and $c_1 \geqslant 3$, which is a contradiction. So rational functions satisfying (2.7) do not exist.

Proofs of Theorems 2.1 and 2.3.

Proof of Theorem 2.1. *Step* 1. We start with the case $i = 0$. First, the function x defined by (2.1) is increasing on $[0, 1]$. Moreover, x' is convex increasing on $[0, \frac{1}{2}]$ and convex decreasing on $[\frac{1}{2}, 1]$. In addition,

$$x'(\theta) \leqslant 2(2p+1)(1-x^2)^{2p/2p+1}.$$

Because of the symmetry of the points, we only need to prove the theorem for $x > 0$ and $x \neq x_k$, $k = 0, ..., n$. Hence $x = x(\theta)$, $\theta \in [\frac{1}{2}, 1]$, $x_{d-1} < x < x_d$ and we may assume that x_d is the closest point to x, i.e., $|x_d - x| \leqslant |x - x_{d-1}|$. It follows easily that

$$|x - x_d| = \left| \int_\theta^{d/n} x'(u)\, du \right| \leqslant x'(\theta)\, n^{-1}$$

since x' is decreasing on $[\frac{1}{2}, 1]$. From the defining of $S^s_{n,r} f$ we get

$$f(x) - S^s_{n,r}(X; f; x) = \frac{\sum_{k=0}^{n} |x - x_k|^{-s} \left[f(x) - \sum_{j=0}^{r} f^{(j)}(x_k)/j!\,(x - x_k)^j\right]}{\sum_{k=0}^{n} |x - x_k|^{-s}}.$$

Since

$$\left| f(x) - \sum_{j=0}^{r} \frac{f^{(j)}(x_k)}{j!}(x - x_k)^j \right| \leqslant |x - x_k|^r\, \omega(f^{(r)}; |x - x_k|)$$

and

$$\sum_{k=0}^{n} |x - x_k|^{-s} \geqslant |x - x_d|^{-s},$$

it follows that

$$\begin{aligned} &|f(x) - S^s_{n,r}(X; f; x)| \\ &\quad \leqslant |x - x_d|^r \sum_{k=0}^{n} \left[\frac{|x - x_d|}{|x - x_k|}\right]^{s-r} \omega(f^{(r)}; |x - x_k|) \\ &\quad \leqslant [x'(\theta)\, n^{-1}]^r \sum_{k=0}^{n} \left[\frac{|x - x_d|}{|x - x_k|}\right]^{s-r} \\ &\qquad \times \omega(f^{(r)}; |x - x_k|) := [x'(\theta)\, n^{-1}]^r\, \Sigma_m(x). \end{aligned}$$

Now we are going to estimate $\Sigma_m(x)$. Since

$$\begin{aligned} \Sigma_m(x) &\leqslant \omega(f^{(r)}; |x - x_d|) \\ &\quad + \left[\frac{|x - x_d|}{|x - x_{d-1}|}\right]^{s-r} \omega(f^{(r)}; |x - x_{d-1}|) \\ &\quad + \Sigma^1_{x_k \leqslant 0} + \Sigma^2_{0 < x_k \leqslant x_{d-2}} \\ &\quad + \Sigma^3_{x_{d+1} \leqslant x_k \leqslant 1} \left[\frac{|x - x_d|}{|x - x_k|}\right]^{s-r} \omega(f^{(r)}; |x - x_k|), \end{aligned}$$

using the obvious bounds for the terms corresponding to $k = d$ and $d - 1$, we obtain

$$\begin{aligned} \Sigma_m(x) &\leqslant +3\omega(f^{(r)}; x'(\theta)\, n^{-1}) + \Sigma^1_{x_k \leqslant 0} + \Sigma^2_{0 < x_k \leqslant x_{d-2}} \\ &\quad + \Sigma^3_{x_{d+1} \leqslant x_k \leqslant 1} \left[\frac{|x - x_d|}{|x - x_k|}\right]^{s-r} \omega(f^{(r)}; |x - x_k|). \end{aligned}$$

To estimate Σ^2, we observe that

$$|x - x_k| = \left|\int_{k/n}^{\theta} x'(u)\,du\right| \geqslant x'(\theta)(d-1-k)\,n^{-1},$$

$$k = [n/2]+1, \ldots, d-2$$

and

$$\frac{\omega(f^{(r)}; |x-x_k|)}{|x-x_k|} \leqslant 2\,\frac{\omega(f^{(r)}; x'(\theta)(d-1-k)\,n^{-1})}{x'(\theta)(d-1-k)\,n^{-1}}.$$

Therefore,

$$\Sigma^2 \leqslant 2 \sum_{k=[n/2]+1}^{d-2} \frac{\omega(f^{(r)}; x'(\theta)(d-k-1)\,n^{-1})}{(d-k-1)^{s-r}}$$

and, by standard calculations, one has

$$\Sigma^2 \leqslant \frac{A}{n^{s-r-1}} \int_{1/n}^{1} \omega(f^{(r)}; t(1-x^2)^{2p/2p+1})\,\frac{dt}{t^{s-r}}$$

with the constant A depending on p, r, and α. To estimate Σ^3, we note that

$$|x - x_k| = \left|\int_{\theta}^{k/n} x'(u)\,du\right| > \int_{\theta}^{(\theta+k/n)/2} x'(u)\,du$$

and, since x' is decreasing,

$$|x-x_k| \geqslant \frac{k/n-\theta}{2}\,x'\left(\frac{k/n+\theta}{2}\right) > \frac{k/n-\theta}{2}\,x'\left(\frac{\theta+1}{2}\right)$$

$$= (k/n-\theta)\,2^{-2p-1}x'(\theta).$$

Hence,

$$\Sigma^3 \leqslant 2[2^{2p+1}]^{s-r} \sum_{k=d+1}^{n} \omega(f^{(r)}; x'(\theta)(k-d)\,n^{-1})(k-d)^{r-s}$$

$$\leqslant \frac{A}{n^{s-r-1}} \int_{1/n}^{1} \omega(f^{(r)}; t(1-x^2)^{2p/(2p+1)})\,\frac{dt}{t^{s-r}}.$$

Now it remains to estimate Σ^1. Because of obvious symmetry reasons, this can be done as in the case of the sums Σ^2 and Σ^3.

Finally,

$$\Sigma_m(x) \leqslant \frac{A}{n^{s-r-1}} \int_{1/n}^{1} \omega(f^{(r)}; t(1-x^2)^{2p/(2p+1)})\,\frac{dt}{t^{s-r}} \tag{3.1}$$

and, therefore, (2.2) is true for $i=0$.

Step 2. Now assume $i>0$. Let

$$A_k(x)=\frac{|x-x_k|^{-s}}{\sum_{k=0}^{n}|x-x_k|^{-s}}$$

and

$$T_{r,k}(f;x)=\sum_{j=0}^{r}\frac{f^{(j)}(x_k)}{j!}(x-x_k)^j.$$

With this notation, we get

$$[f(x)-S_{n,r}(f;x)]^{(i)}=\sum_{l=0}^{i}\binom{i}{l}\sum_{k=0}^{n}A_k^{(l)}(x)[f(x)-T_{r,k}(f;x)]^{(i-l)}.$$

Since

$$|[f(x)-T_{r,k}(f;x)]^{(i-l)}|\leqslant|x-x_k|^{r-i+l}\,\omega(f^{(r)};|x-x_k|),$$

we obtain the inequality

$$\begin{aligned}&|[f(x)-S_{n,r}(f;x)]^{(i)}|\\&\quad\leqslant\sum_{l=0}^{i}\binom{i}{l}\sum_{k=0}^{n}|A_k^{(l)}(x)|\,|x-x_k|^{r-i+l}\,\omega(f^{(r)};|x-x_k|).\end{aligned}\tag{3.2}$$

Now we introduce the functions

$$g_q(x)=\frac{\sum_{k=0}^{n}|x-x_k|^{-s-q}}{\sum_{k=0}^{n}|x-x_k|^{-s}},\qquad q\in N.$$

We have

$$|g_q(x)|\leqslant|x-x_d|^{-q},\qquad |g_q(x)\,g_p(x)|\leqslant|x-x_d|^{-q-p},\tag{3.3}$$

and

$$|g_q^{(j)}(x)|\leqslant\text{const}\,|x-x_d|^{-q-j},\tag{3.4}$$

where, as before, x_d is the closest point to x. Hence

$$|A_k'(x)|\leqslant sA_k(x)[g_1(x)+|x-x_k|^{-1}],$$

and, therefore,

$$|A_k'(x)|\leqslant 2sA_k(x)\,|x-x_d|^{-1}.$$

Moreover,

$$\begin{aligned}|A_k''(x)| &\leqslant s\,|A_k'(x)|\,[g_1(x)+|x-x_k|^{-1}]\\ &\quad + sA_k(x)[g_1'(x)+|x-x_k|^{-2}]\\ &\leqslant \text{const}\,A_k(x)\,|x-x_d|^{-2}.\end{aligned}$$

Iterating this procedure, by (3.3) and (3.4) we get

$$|A_k^{(l)}(x)| \leqslant \text{const}\,A_k(x)\,|x-x_d|^{-l}.$$

Thus, from (3.2),

$$\begin{aligned}&|[f(x)-S_{n,r}^{s}(f;x)]^{(i)}|\\ &\quad\leqslant \text{const}\sum_{k=0}^{n} A_k(x)\,|x-x_d|^{-l}\,|x-x_k|^{r-i+l}\,\omega(f^{(r)};|x-x_k|)\\ &\quad\leqslant \text{const}\sum_{k=0}^{n} A_k(x)\,|x-x_d|^{-i}\,|x-x_k|^{r}\,\omega(f^{(r)};|x-x_k|)\\ &\quad\leqslant \text{const}\,|x-x_d|^{r-i}\sum_{k=0}^{n}\left[\frac{|x-x_d|}{|x-x_k|}\right]^{s-r}\omega(f^{(r)};|x-x_k|)\\ &\quad\leqslant \text{const}[x'(\theta)\,n^{-1}]^{r-i}\,\Sigma_m(x).\end{aligned}$$

From (3.1) the assertion follows, if x_d is the closest point to x. If x_{d-1} is the closest point to x, then the above proof still holds with an obvious minor modification. ∎

Proof of Theorem 2.3. We start with the case $i=0$. Given $p=1,2,\ldots$, we introduce the function $z=z_p\colon[0,1]\to[-1,1]$ defined by $z(\theta)=(2\theta-1)^{2p+1}$. Obviously, $z_k=z_{n,k}=z(k/n)$, $k=0,\ldots,n$. Note that z is increasing on $[0,1]$. Moreover, z' is convex decreasing on $[0,\frac{1}{2}]$ and convex increasing on $[\frac{1}{2},1]$. In addition,

$$z'(\theta)\leqslant 2(2p+1)\,|x|^{2p/(2p+1)}.$$

Because of the symmetry of the points we only need to prove the theorem for $x<0$ and $x\neq x_k$, $k=0,\ldots,n$. Hence $x=z(\theta)$, $\theta\in[0,\frac{1}{2}]$, $x_{d-1}<x<x_d$, and we may assume, again, that x_d is the closest point to x, i.e., $|x_d-x|\leqslant|x-x_{d-1}|$. Obviously,

$$|x-x_d|=\left|\int_{\theta}^{d/n} z'(u)\,du\right|\leqslant z'(\theta)\,n^{-1}.$$

From the definition of $S^s_{n,r}f$ we get

$$|f(x) - S^s_{n,r}(Z;f;x)| \leqslant \frac{\sum_{k=0}^{n} |x-x_k|^{r-s}\, \omega(f^{(r)}; |x-x_k|)}{\sum_{k=0}^{n} |x-x_k|^{-s}}$$

$$\leqslant |x-x_d|^r \sum_{k=0}^{n} \left[\frac{|x-x_d|}{|x-x_k|}\right]^{s-r} \omega(f^{(r)}; |x-x_k|)$$

$$\leqslant [z'(\theta)\, n^{-1}]^r\, \Sigma_m(x).$$

Now the proof proceeds exactly as in the previous one. For $i > 0$ the proof is analogous; and (2.4) follows. ∎

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